Topology and its Applications 217 (2017) 38–50

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Variations of selective separability and tightness in function spaces with set-open topologies ☆

Alexander V. Osipov [a], Selma Özçağ [b,*]

[a] Ural Federal University, Krasovskii Institute of Mathematics and Mechanics, Ekaterinburg, Russia
[b] Department of Mathematics, Hacettepe University, Beytepe, Ankara, Turkey

ARTICLE INFO

Article history:
Received 25 July 2016
Received in revised form 7 December 2016
Accepted 13 December 2016
Available online 15 December 2016

MSC:
54C35
54C40
54D65
46E55

Keywords:
Selection principles
Bitopology
Selective separability
Set-open topology
C-compactness
Submetrizable
Fan tightness
Strong fan tightness
T-tightness
R-separability
M-separability
GN-separability
H-separability

ABSTRACT

We study tightness properties and selective versions of separability in bitopological function spaces endowed with set-open topologies.

1. Introduction

In this paper we are mainly concerned with selective versions of separability in bitopological function spaces endowed with two homogenous set-open topologies.

☆ This work was supported by Act 211 Government of the Russian Federation, contract 02.A03.21.0006.
* Corresponding author.
E-mail addresses: OAB@list.ru (A.V. Osipov), sozcag@hacettepe.edu.tr (S. Özçağ).

http://dx.doi.org/10.1016/j.topol.2016.12.010
0166-8641/

Variations of separability, stronger forms, weaker forms, functional separability and similar properties have been intensively studied by many mathematicians in the last two decades. The selective versions of separability have recently gained a particular attention and many interesting results were obtained.

Although the definition of selection principles was given by Scheepers in 1996, the theory is actually based on the papers by Menger, Hurewicz, Rothberger and Sierpiński in 1920–1930, see [11,17,25].

Many topological properties can be defined or characterized in terms of the following two classical selection principles given in a general form in [28] as follows:

Let $\mathcal{A}$ and $\mathcal{B}$ be sets consisting of families of subsets of an infinite set X. Then:

$S_1(\mathcal{A}, \mathcal{B})$: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(b_n : n \in \mathbb{N})$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$S_{fin}(\mathcal{A}, \mathcal{B})$: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each n, $B_n \subseteq A_n$, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

The selection principles denoted by $S_{fin}(\mathcal{O}, \mathcal{O})$ and $S_1(\mathcal{O}, \mathcal{O})$ are called the Menger and Rothberger property, where $\mathcal{O}$ is the family of open covers of a topological space.

For the topological space X, let $\mathcal{D}$ denote the family of dense subspaces of X. The selection principles $S_{fin}(\mathcal{D}, \mathcal{D})$ and $S_1(\mathcal{D}, \mathcal{D})$ were introduced by Scheepers in [29] and recently gained a great attention, see [3–5, 10].

In [3] the selection properties $S_{fin}(\mathcal{D}, \mathcal{D})$, $S_1(\mathcal{D}, \mathcal{D})$ and $S_1(\mathcal{D}, \mathcal{D}^{gp})$ are called M-separability (also called selective separability), R-separability and GN-separability, respectively, while a modified property $S_{fin}(\mathcal{D}, \mathcal{D}^{gp})$ is called H-separability where "M-", "R-" and "H-" represent well known Menger, Rothberger and Hurewicz properties.

It should be noted that very recently Tsaban and his co-authors in [6] studied all properties $S(\mathcal{A}, \mathcal{B})$ for $S \in \{S_1, S_{fin}\}$ and $\mathcal{A}, \mathcal{B}$ are combinations of open covers, dense open families and dense sets.

The selection principle theory was first considered in bitopological spaces by Kočinac and Özçağ in [15, 16] and they carried out a systematic study on selection principles mainly selective versions of separability in bitopological spaces, particularly in the space $C(X)$ of all continuous real-valued functions defined on a Tychonoff space X, where $C(X)$ is endowed with the topology τ_p of pointwise topology and the compact-open topology τ_k.

In the present paper we investigate some properties of bitopological selective versions of separability in function spaces and the set-open topologies will be the main tool.

The set-open topology on a family λ of nonempty subsets of a set X is a generalization of the compact-open topology (and of the topology of pointwise convergence). This notion was first introduced by Arens and Dugundji in [1] and was widely investigated by Osipov in [20–22]. In the next section we recall some facts on the set-open topologies.

For background material on selection principles we refer to the survey papers [13,27,30], for undefined notions in function spaces, see [2]. We will follow [8] for topological terminology and notations.

2. Main definitions and notations

In this paper, we consider the space $C(X)$ of all real-valued continuous functions defined on a Tychonoff space X.

Recall that a subset A of a space X is a *C-compact subset of X* if for any real-valued function f continuous on X, the set $f(A)$ is compact in $\mathbb{R}$.

A family λ of C-compact subsets of X is said to be hereditary with respect to C-compact subsets if it satisfies the following condition: whenever $A \in \lambda$ and B is a C-compact (in X) subset of A, then $B \in \lambda$. Recall that a family λ of nonempty subsets of a topological space (X, τ) is called a *π-network* for X if for any nonempty open set $U \in \tau$, there exists $A \in \lambda$ such that $A \subseteq U$.

Let $\Psi = \{\lambda : \lambda$ is a family of C-compact subsets of X which hereditary with respect to C-compact subsets and λ is π-network for $X\}$. Note that if p (k) is a set all finite (compact) subsets of X, then $p \in \Psi$ ($k \in \Psi$).

For a family λ of nonempty subsets of a topological space X the element of the standard subbase of the λ-open topology:

$$[F, U] = \{f \in C(X) : \ f(F) \subseteq U\},$$

where $F \in \lambda$ and U is an open subset of $\mathbb{R}$.

We use the following notation for various topological spaces with the underlying set $C(X)$ and $\lambda, \mu \in \Psi$:

$C_\lambda(X)$ or $(C(X), \tau_\lambda)$ for the λ-open topology;

$(C(X), \tau_\lambda, \tau_\mu)$ for bitopological space $C(X)$ endowed with two topologies τ_λ and τ_μ.

Given a family λ of nonempty subsets of X, let $\lambda(C) = \{A \in \lambda$: for every C-compact subset B of the space X with $B \subset A$, the set $[B, U]$ is open in $C_\lambda(X)$ for any open set U of the space $\mathbb{R}\}$. Let λ_m denote the maximal with respect to inclusion family, provided that $C_{\lambda_m}(X) = C_\lambda(X)$. Note that a family λ_m is unique for each family λ.

Interest in studying the λ-open topology is generated by Theorem 3.3 in [20] which characterizes some topological-algebraic properties of the set-open topology.

The following theorem is a corollary of Theorem 3.3 in [20].

Theorem 2.1. *For a space X, the following statements are equivalent:*

1. *$C_\lambda(X)$ is a paratopological group;*
2. *$C_\lambda(X)$ is a topological group;*
3. *$C_\lambda(X)$ is a topological vector space;*
4. *$C_\lambda(X)$ is a locally convex topological vector space;*
5. *$C_\lambda(X)$ is a topological ring;*
6. *$C_\lambda(X)$ is a topological algebra;*
7. *λ is a family of C-compact sets and $\lambda = \lambda(C)$;*
8. *λ_m is a family of C-compact sets and it is hereditary with respect to C-compact subsets.*

Without loss of generality we can assume that $C_\lambda(X)$ is a paratopological group (TVS, locally convex TVS) under the usual operations of addition and multiplication (and multiplication by scalars) if and only if $\lambda_m \in \Psi$.

Further, throughout the article, we assume $\lambda = \lambda_m \in \Psi$.

In particular, if $\lambda = p$ ($\lambda = k$) then $C_\lambda(X) = C_p(X)$ ($C_\lambda(X) = C_k(X)$), i.e. the λ-open topology coincide with the topology of pointwise convergence (the compact-open topology).

Since $C_\lambda(X)$ is a homogenous space, we may always consider the point $\mathbf{0}$ when studying local properties of this space.

We use the symbol Ω_x to denote the set $\{A \subset X : x \in Cl_\tau(A) \setminus A\}$, where (X, τ) is a topological space and $x \in X$.

Since we are considering two topologies of $C(X)$ we shall use the symbol $(\Omega_0)^\lambda$ to denote Ω_0 in the space $C_\lambda(X)$ and the symbol $(\Omega_0)^\mu$ to denote Ω_0 in the space $C_\mu(X)$ where $\lambda, \mu \in \Psi$.

We will denote by τ_λ the λ-open topology on $C(X)$ for $\lambda \in \Psi$.

Before we proceed let us recall some definitions, notations and terminology which were used in [15,16].

Throughout this paper (X, τ_1, τ_2) will be a bitopological space (shortly bispace), i.e. the set X endowed with two topologies τ_1 and τ_2. For a subset A of X, Cl_i will denote the closure of A in (X, τ_i), $i = 1, 2$.

A subset A of X is bidense (shortly d-dense) in (X,τ_1,τ_2) if A is dense in both (X,τ_1) and (X,τ_2). X is d-separable if there is a countable set A which is d-dense in (X,τ_1,τ_2).

Denote by $\mathcal{D}_1$ and $\mathcal{D}_2$ the collections of all dense subsets of (X,τ_1) and (X,τ_2), respectively. We say that X is:

M_{τ_i,τ_j}*-separable* $(i,j=1,2; i\neq j)$, if for each sequence $(D_n : n\in\mathbb{N})$ of elements of $\mathcal{D}_i$ there are finite sets $F_n\subset D_n$, $n\in\mathbb{N}$, such that $\bigcup_{n\in\mathbb{N}} F_n\in\mathcal{D}_j$, i.e. if $S_{fin}(\mathcal{D}_i,\mathcal{D}_j)$ holds;

R_{τ_i,τ_j}*-separable* if $S_1(\mathcal{D}_i,\mathcal{D}_j)$ holds;

H_{τ_i,τ_j}*-separable* if for each sequence $(D_n : n\in\mathbb{N})$ of elements of $\mathcal{D}_i$ there are finite sets $F_n\subset D_n$, $n\in\mathbb{N}$, such that each τ_j-open subset of X intersects F_n for all but finitely many n;

GN_{τ_i,τ_j}*-separable* if $S_1(\mathcal{D}_i,\mathcal{D}_j^{gp})$ holds.

Here $\mathcal{D}_j^{gp}$ is the collection of groupable dense subsets of a space; a countable dense subset D of a space Z is *groupable* if $D=\bigcup_{n\in\mathbb{N}} A_n$, each A_n finite and each open set U in Z intersects all but finitely many A_n [9].

In case $\tau_1=\tau_2=\tau$, these definitions coincide with definitions of corresponding topological selective versions of separability of (X,τ).

As mentioned in [15] we have the implications, GN_{τ_i,τ_j}-separable $\Longrightarrow$ R_{τ_i,τ_j}-separable $\Longrightarrow$ M_{τ_i,τ_j}-separable, and H_{τ_i,τ_j}-separable $\Longrightarrow$ M_{τ_i,τ_j}-separable.

The remaining notations can be found in [8,15,16].

3. The tightness-type properties

In this section we give some results on bitopological versions of the tightness properties and its variations in function bispaces. We also combine these results with bitopological selective separability properties.

In analogy to the (τ_i,τ_j)-tightness in [15], (τ_λ,τ_μ)-tightness is introduced by replacing τ_i and τ_j with τ_λ and τ_μ topologies respectively.

The (τ_λ,τ_μ)*-tightness*, $\lambda,\mu\in\Psi$, of a bispace $(C(X),\tau_\lambda,\tau_\mu)$ is the least infinite cardinal κ such that whenever $A\subseteq C(X)$ and $f\in Cl_{\tau_\lambda}(A)$, there is $B\subseteq A$ such that $|B|\leq\kappa$ and $f\in Cl_{\tau_\mu}(B)$.

We recall that a subset A of X is called co-zero (or a functional open) set if $X\setminus A$ is a zero set. We mean by a zero set, a subset of X that is complete preimage of zero for certain function from $C(X)$.

Definition 3.1. A co-zero (functional open) family $\mathbb{U}$ of X is called a λ*–*f*-cover* if X is not a member of $\mathbb{U}$ and for each $A\in\lambda$ there is a $U\in\mathbb{U}$ such that $A\subseteq U$.

Note that a λ–f-cover is a cover of $\bigcup\lambda$, but may fail to be a cover of X. The symbol $\Lambda(\lambda)$ denotes the collection of all λ–f-covers for the family λ.

Definition 3.2. A space X is λ*–*μ*-Lindelöf* if for each $\mathcal{U}\in\Lambda(\lambda)$ there is a $\mathcal{V}\subseteq\mathcal{U}$ such that $\mathcal{V}$ is countable and $\mathcal{V}\in\Lambda(\mu)$.

If $\lambda=\mu$ then we shall write it simply λ-Lindelöf. Note that if $\lambda=k$ then λ-Lindelöf is k-Lindelöf.

Theorem 3.3. *For a space X and λ, $\mu\in\Psi$ with $\mu\subseteq\lambda$, the bispace $(C(X),\tau_\lambda,\tau_\mu)$ has countable (τ_λ,τ_μ)-tightness if and only if X is λ–μ-Lindelöf.*

Proof. $(\Rightarrow)$. Suppose $(C(X),\tau_\lambda,\tau_\mu)$ has countable (τ_λ,τ_μ)-tightness and $\mathcal{U}\in\Lambda(\lambda)$. For each pair (K,U), $K\in\lambda$, $U\in\mathcal{U}$ and $K\subseteq U$, let $f_{K,U}$ be any continuous function from X to $[0,1]$, such that $f_{K,U}(K)\subseteq\{0\}$ and $f_{K,U}(X\setminus U)\subseteq\{1\}$. Let $A=\{f_{K,U} : K\in\lambda, K\subseteq U\in\mathcal{U}\}$. Then $\mathbf{0}$ belongs to the closure of A with respect to the τ_λ topology. Since $(C(X),\tau_\lambda,\tau_\mu)$ has countable (τ_λ,τ_μ)-tightness there is a countable set $B=\{f_{K_n,U_n} : n\in\mathbb{N}\}$, such that $\mathbf{0}$ belongs to the closure of B with respect to the τ_μ topology. We claim that $\{U_n : n \in \mathbb{N}\} \in \Lambda(\mu)$. Let $F \in \mu$. From the fact that $\mathbf{0}$ belongs to the closure of B with respect to the τ_μ topology, it follows that there is $i \in \mathbb{N}$, such that $[F, (-1, 1)]$ contains the function f_{K_i,U_i}. Then $F \subseteq U_i$. Overwise for some $x \in F$ one has $x \notin U_i$ so that $f_{K_i,U_i}(x) = 1$, contradicting $f_{K_i,U_i} \in [F, (-1, 1)]$.

($\Leftarrow$). Let A be a set of $C(X) \setminus \{\mathbf{0}\}$, such that its closure with respect to the τ_λ topology contains $\mathbf{0}$. If $\{X\} \in \lambda$ (X is pseudocompact), then the τ_λ topology coincides with the C-compact-open topology, so $C_\lambda(X)$ is metrizable (Theorem 2.2 in [22]), thus first countable, which means that we can find a sequence $(a_n : n \in \mathbb{N})$, converging uniformly to $\mathbf{0}$, so there is nothing to be proved.

Let $\{X\} \notin \lambda$. For each $n \in \mathbb{N}$ and every set $K \in \lambda$, the neighborhood $[K, (-1/n, 1/n)]$ of $\mathbf{0}$ intersects A, so there exists a continuous function $f_{K,n} \in A$, such that $|f_{K,n}(x)| < 1/n$ for each $x \in K$. Since $f_{K,n}$ is a continuous function there is a co-zero set $U_{K,n}$, such that $f_{K,n}(U_{K,n}) \subseteq (-1/n, 1/n)$. Let $\mathcal{U}_n = \{U_{K,n} : K \in \lambda\}$.

As for any subset $K \in \lambda$ we have $K \neq X$, it can easily be achieved that none of the sets $U_{K,n}$ above coincides with X. So for each n, $\mathcal{U}_n \in \Lambda(\lambda)$. Each $\mathcal{U}_n$ has countable μ–f-cover $\mathcal{V}_n \subseteq \mathcal{U}_n$. Define $B = \{f_{K,n} : n \in \mathbb{N}, U_{K,n} \in \mathcal{V}_n\}$. It is clear that $B \subseteq A$, $|B| \leq \aleph_0$, and $\mathbf{0}$ belongs to the closure of B with respect to the τ_μ topology. Therefore the bispace $(C(X), \tau_\lambda, \tau_\mu)$ has countable (τ_λ, τ_μ)-tightness. □

Corollary 3.4. *The space $C_\lambda(X)$ has countable tightness if and only if X is λ-Lindelöf.*

Definition 3.5. A space X has *countable (τ_λ, τ_μ)-fan tightness* if for each $x \in X$ and each sequence $(A_n : n \in \mathbb{N})$ of elements of $(\Omega_x)^\lambda$ there exists a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that, for each n, $B_n \subseteq A_n$ and $x \in cl_{\tau_\mu}(\bigcup_{n\in\mathbb{N}} B_n)$, i.e. if $S_{fin}((\Omega_x)^\lambda, (\Omega_x)^\mu)$ holds for each $x \in X$.

Definition 3.6. A space X has *countable (τ_λ, τ_μ)-strong fan tightness* if for each $x \in X$, $S_1((\Omega_x)^\lambda, (\Omega_x)^\mu))$ holds.

The proof of the next theorem follows the lines of the proof of Theorem 2.4 in [23].

Theorem 3.7. *Let X be a Tychonoff space, λ, $\mu \in \Psi$ and $\mu \subseteq \lambda$. Then the following are equivalent:*

1. *$C(X)$ satisfies $S_1((\Omega_0)^\lambda, (\Omega_0)^\mu))$;*
2. *X has property $S_1(\Lambda(\lambda), \Lambda(\mu))$.*

Proof. (1) $\Rightarrow$ (2). Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of λ–f covers of X. For each pair $A \in \lambda$ and a co-zero set U such that $A \subseteq U$, let $f_{A,U} : X \to [0, 1]$ be a continuous function with $f_{A,U}(A) \subseteq \{0\}$ and $f_{A,U}(X \setminus U) \subseteq \{1\}$.

Now, let $B_n = \{f_{A,U} : A \in \lambda, A \subseteq U \in \mathcal{U}_n\}$. We claim that $\mathbf{0}$ is in the closure of each B_n, with respect to the λ-open topology.

Indeed, at first we have $\mathbf{0} \notin B_n$. Let $\mathbf{0} \in [A, V]$, where $A \in \lambda$ and V is an open subset of $\mathbb{R}$. There exists $U \in \mathcal{U}_n$ with $A \subseteq U$. The function $f_{A,U} \in B_n$ belongs to $[A, V]$ and $f_{A,U}(A) = \{0\} \subseteq V$, hence $\mathbf{0}$ is in the closure of B_n with respect to the λ-open topology, which means $B_n \in (\Omega_0)^\lambda$.

Now, since $C(X)$ has countable (τ_λ, τ_μ) strong fan tightness, there is a sequence $(f_{A_n,U_n} : n \in \mathbb{N})$ with $A_n \in \lambda$, $U_n \in \mathcal{U}_n$, such that $\mathbf{0}$ belongs to the closure of $\{f_{A_n,U_n} : n \in \mathbb{N}\}$ with respect to the μ-open topology. To conclude the first part of the proof we claim that $\{U_n : n \in \mathbb{N}\} \in \Lambda(\mu)$. Let $K \in \mu$. There is $j \in \mathbb{N}$, such that $[K, (-1, 1)]$ contains the function $f_{A_j,Uj}$. Clearly $K \subseteq U_j$.

(2) $\Rightarrow$ (1). Let $(B_m : m \in \mathbb{N}) \in {\Omega_0}^\lambda$. If $\{X\} \in \lambda$ there is nothing to prove. Let $\{X\} \notin \lambda$. Given the bijection $j : \mathbb{N}^2 \mapsto \mathbb{N}$, we set $B_{n,m} := B_{j(n,m)}$. For each $A \in \lambda$ the neighborhood $[A, (-1/n, 1/n)]$ of $\mathbf{0}$ intersects $B_{m,n}$, which means that there exists a continuous function $f_{A,m,n} \in B_{m,n}$, such that $|f_{A,m,n}(x)| < 1/n$ for each $x \in A$ and $f_{A,m,n}(U_{A,m,n}) \subseteq (-1/n, 1/n)$ for the co-zero set $U_{A,m,n}$.

Now we set $\mathcal{U}_{m,n} = \{U_{A,m,n}, A \in \lambda\}$. For any set $A \in \lambda$, $A \neq X$, none of the sets $U_{A,m,n}$ equals to X. So for each m and n, $\mathcal{U}_{m,n} \in \Lambda(\lambda)$. To each sequence $(\mathcal{U}_{m,n} : m \in \mathbb{N})$ apply the fact that X is an

$S_1(\Lambda(\lambda), \Lambda(\mu))$-space. We can easily obtain a sequence $(U_{A_{m,n}} : m \in \mathbb{N})$, such that $\{U_{A_{m,n}} : m \in \mathbb{N}\} \in \Lambda(\mu)$. Now define $D = \{f_{A_{m,n}} : n \in \mathbb{N}, U_{A,m,n} \in \{U_{A_{m,n}} : m \in \mathbb{N}\}\}$. It is evident that $D \subseteq B_n$, $|D| \leq \aleph_0$, and $\mathbf{0}$ belongs to the closure of D with respect to the τ_μ topology. Therefore the bispace $(C(X), \tau_\lambda, \tau_\mu)$ satisfies $S_1((\Omega_0)^\lambda, (\Omega_0)^\mu))$. $\square$

Corollary 3.8. *For a Tychonoff space X the following are equivalent:*

1. *$C_\lambda(X)$ has countable strong fan tightness;*
2. *X has property $S_1(\Lambda(\lambda), \Lambda(\lambda))$.*

In a similar way one can show

Theorem 3.9. *Let X be a Tychonoff space, λ, $\mu \in \Psi$ and $\mu \subseteq \lambda$. Then the following are equivalent:*

1. *$C(X)$ satisfies $S_{fin}((\Omega_0)^\lambda, (\Omega_0)^\mu))$;*
2. *X has property $S_{fin}(\Lambda(\lambda), \Lambda(\mu))$.*

Corollary 3.10. *The space $C_\lambda(X)$ has countable fan tightness if and only if X has the property $S_{fin}(\Lambda(\lambda), \Lambda(\lambda))$.*

The T-tightness $T(X)$ of a space X is the smallest infinite cardinal τ such that if $\{F_\alpha : \alpha < \kappa\}$ is an increasing family of closed subsets and $cf(\kappa) > \tau$, then $\bigcup\{F_\alpha : \alpha < \kappa\}$ is closed in X. This definition was introduced in [19] by Juhász. Since the family $\{F_\alpha : \alpha < \kappa\}$ is increasing and $cf(\kappa)$ is regular, we may say that the T-tightness $T(X)$ is the smallest infinite cardinal τ such that if $\{F_\alpha : \alpha < \kappa\}$ is an increasing family of closed subsets and κ is a regular cardinal greater than τ, then $\bigcup\{F_\alpha : \alpha < \kappa\}$ is closed in X. The T-tightness of function space $C_p(X)$ (resp. $C_k(X)$) was investigated in [26] (resp. in [12]).

The bitopological version of this notion was introduced in [16].

A bispace $(X, \tau_\mu, \tau_\lambda)$ has *countable (τ_μ, τ_λ)–T-tightness*, if for each uncountable regular cardinal κ and each increasing sequence $(A_\alpha : \alpha < \kappa)$ of closed subsets of (X, τ_μ), the set $\bigcup\{A_\alpha : \alpha < \kappa\}$ is closed in (X, τ_λ).

Theorem 3.11. *Let X be a Tychonoff space, λ, $\mu \in \Psi$ and $\mu \subseteq \lambda$. Then* (1) *implies* (2)*:*

1. *$(C(X), \tau_\mu, \tau_\lambda)$ has countable (τ_λ, τ_μ)–T-tightness;*
2. *for each regular cardinal κ and each increasing sequence $(\mathcal{U}_\alpha : \alpha < \kappa)$ of family of cozero subsets of X such that $\bigcup_{\alpha<\kappa} \mathcal{U}_\alpha$ is a λ–f-cover of X, there is a $\beta < \kappa$ so that $\mathcal{U}_\beta$ is a μ–f-cover of X.*

Proof. Since a bispace $(C(X), \tau_\mu, \tau_\lambda)$ has countable (τ_λ, τ_μ)–T-tightness, for each uncountable regular cardinal κ and each increasing sequence $(A_\alpha : \alpha < \kappa)$ of closed subsets of $(C(X), \tau_\lambda)$, the set $\bigcup\{A_\alpha : \alpha < \kappa\}$ is closed in $(C(X), \tau_\mu)$. Let for regular cardinal κ and increasing sequence $(\mathcal{U}_\alpha : \alpha < \rho)$ of family of cozero subsets of X, the family $\bigcup_{\alpha<\kappa} \mathcal{U}_\alpha$ be a λ–f-cover of X. Now we set $\mathcal{U}_{\alpha,K} := \{U \in \mathcal{U}_\alpha : K \subset U\}$ for each $\alpha < \kappa$ and $K \in \lambda$. For each $U \in \mathcal{U}_{\alpha,K}$, let $f_{K,U}$ be a continuous function from X into $[0, 1]$, such that $f_{K,U}(K) = \{0\}$ and $f_{K,U}(X \setminus U) = \{1\}$. Now consider the set $A_\alpha = \{f_{K,U} : U \in \mathcal{U}_{\alpha,K}\}$, $\alpha < \kappa$.

By (1) we observe that the set $A = \bigcup_{\alpha<\kappa} cl_{\tau_\lambda}(A_\alpha)$ is closed in $(C(X), \tau_\mu)$.

Let $< \mathbf{0}, K, \varepsilon >:= [K, (-\varepsilon, \varepsilon)]$ be a standard basic τ_λ-neighborhood of $\mathbf{0}$. There exist $\alpha < \kappa$ and $U \in \mathcal{U}_\alpha$ with $K \subset U$. Then $U \in \mathcal{U}_{\alpha,K}$, hence by construction there is $f \in A_\alpha \bigcap < \mathbf{0}, K, \varepsilon >$. Therefore, each τ_λ-neighborhood of $\mathbf{0}$ intersects some A_α, $\alpha < \kappa$, i.e. $\mathbf{0}$ belongs to the τ_λ-closure of the set $\bigcup_{\alpha<\kappa} A_\alpha$ which is actually the set A. It follows that there is $\beta < \kappa$ with $\mathbf{0} \in cl_{\tau_\mu}(A_\beta)$. We claim that the corresponding family $\mathcal{U}_\beta$ is an μ–f-cover of X.

Let $F \in \mu$. Then the τ_μ-neighborhood $< \mathbf{0}, F, 1 >$ of $\mathbf{0}$ intersects A_β; let $f_{F,U} \in A_\beta \bigcap < \mathbf{0}, F, 1 >$. Then $f_{F,U}(X \setminus U) = 1$ and thus $F \subset U \in \mathcal{U}_\beta$. Then $\mathcal{U}_\beta$ is an μ–f-cover of X. $\square$

The following example shows that condition (2) may not imply the condition (1).

Example 3.12. Let $X = \omega_1 + 1$ be the space $\{\alpha : \alpha \leq \omega_1\}$ with the order topology, $\mu = p$, $\lambda = k$. Consider bitopological space $(C(X), \tau_p, \tau_k)$. Note that $C_p(X)$ has countable T-tightness (Theorem 2.3 in [26]) and hence for each regular cardinal κ and each increasing sequence $\{\mathcal{U}_\alpha : \alpha < \rho\}$ of family of open subsets of X such that $\bigcup_{\alpha<\kappa} \mathcal{U}_\alpha$ is a ω-cover of X there is a $\beta < \kappa$ so that $\mathcal{U}_\beta$ is an ω-cover of X. Thus $(C(X), \tau_p, \tau_k)$ has property that for each regular cardinal κ and each increasing sequence $\{\mathcal{U}_\alpha : \alpha < \rho\}$ of family of cozero subsets of X such that $\bigcup_{\alpha<\kappa} \mathcal{U}_\alpha$ is a k-cover of X there is a $\beta < \kappa$ so that $\mathcal{U}_\beta$ is an ω-cover of X. Consider the set

$$F_\beta = C(X) \setminus \{h \in C(X) : |h(x)| < \tfrac{1}{2} \text{ for } x \leq \beta \text{ and } |h(x)| < 1 \text{ for } x > \beta\},$$

where β is a limit ordinal in X. Note that F_β is a closed set in $C_k(X)$, but it is not a closed set in $C_p(X)$. So $F = \bigcup_{\beta \leq \omega_1} F_\beta = F_{\omega_1}$. For $\mathbf{0}$ we have $\mathbf{0} \in cl_{\tau_p} F \setminus F$. It follows that bitopological space $(C(X), \tau_p, \tau_k)$ has not countable (τ_k, τ_p)–T-tightness.

Theorem 3.13. *Let X be a Tychonoff space, $\lambda \in \Psi$. Then the following are equivalent:*

1. *$C_\lambda(X)$ has countable T-tightness;*
2. *for each regular cardinal ρ and each increasing sequence $(\mathcal{U}_\alpha : \alpha < \rho)$ of family of cozero subsets of X, such that $\bigcup_{\alpha<\rho} \mathcal{U}_\alpha$ is a λ–f-cover of X, there is a $\beta < \rho$ so that $\mathcal{U}_\beta$ is a λ–f-cover of X.*

Proof. (1) $\Rightarrow$ (2). This follows from Theorem 3.11.

(2) $\Rightarrow$ (1). Conversely, suppose that for each regular cardinal ρ and each increasing sequence $(\mathcal{U}_\alpha : \alpha < \rho)$ of family of cozero subsets of X such that $\bigcup_{\alpha<\rho} \mathcal{U}_\alpha$ is a λ–f-cover of X there is a $\beta < \rho$ so that $\mathcal{U}_\beta$ is a λ–f-cover of X. Let κ be a uncountable regular cardinal and $(A_\alpha : \alpha < \kappa)$ be an increasing sequence of closed subsets of $C_\lambda(X)$.

Suppose $g \in cl_{\tau_\lambda} \bigcup_{\alpha<\kappa} A_\alpha$. Since $C_\lambda(X)$ is homogeneous, we may assume $g = f_0$ where $f_0 := \mathbf{0}$. For every $n \in \mathbb{N}$ and $\alpha < \kappa$, we set $\mathcal{U}_{n,\alpha} = \{f^{-1}(W_n) : f \in A_\alpha\}$, where $W_n = (-\frac{1}{n}, \frac{1}{n})$, and $\mathcal{U}_n = \bigcup_{\alpha<\kappa} \mathcal{U}_{n,\alpha}$. Every $\mathcal{U}_n$ is a λ–f-cover of X.

Indeed, let $F \in \lambda$ and consider the neighborhood $[F, W_n]$ of f_0. By $f_0 \in cl_{\tau_\lambda} \bigcup_{\alpha<\kappa} A_\alpha$, there exist $\alpha < \kappa$ and $f \in A_\alpha \bigcap [F, W_n]$. Then $F \subset f^{-1}(W_n) \in \mathcal{U}_{n,\alpha}$. Thus, every $\mathcal{U}_n$ is a λ–f-cover X. For every $n \in \mathbb{N}$, we can find $\alpha_n < \kappa$ such that $\mathcal{U}_{n,\alpha_n}$ is a λ–f-cover of X. Let γ be the supremum of α_n's. Then for every $n \in \mathbb{N}$, $\mathcal{U}_{n,\gamma}$ is an λ-cover of X. Now we claim $f_0 \in A_\gamma$. Let $[F, W]$ be any neighborhood of f_0 and choose $n \in \mathbb{N}$ with $W_n \subset W$. Since $\mathcal{U}_{n,\gamma}$ is an λ-cover of X, there exists an $f \in A_\gamma$ such that $F \subset f^{-1}(W_n)$. Then $f \in A_\gamma \bigcap [F, W]$. Thus $f_0 \in Cl_{\tau_\lambda}(A_\gamma) = A_\gamma$. We conclude that $\bigcup_{\alpha<\kappa} A_\alpha$ is closed in $C_\lambda(X)$. $\square$

4. Bitopological R-separability and M-separability

R-separability and M-separability in bitopological spaces were first introduced and studied in [15]. We have some analogous results on R-separability and M-separability of bitopological function spaces endowed with the set-open topology.

Definition 4.1. Let X be a topological space and λ be a family of subsets of X. The space X is said to be *separably λ-submetrizable* if there exist a separable metric space Y and a continuous function f from X onto Y such that f is one-to-one on $\bigcup \lambda$.

Note that if $\bigcup \lambda = X$, then X is a submetrizable space.

For example we consider the Mrowka–Isbell space. Let $\mathcal{M}$ be a maximal infinite family of infinite subsets of $\mathbb{N}$ such that the intersection of any two members of $\mathcal{M}$ is finite, and let $\Psi(\mathcal{M}) = \mathbb{N} \bigcup \mathcal{M}$, where a subset U of Ψ is defined to be open provided that for any set $M \in \mathcal{M}$, if $M \in U$ then there is a finite subset F of M such that $\{M\} \bigcup M \setminus F \subset U$.

So, the Mrowka–Isbell space is separably λ-submetrizable space (where λ is a family of finite subsets of the set $\mathbb{N}$ of isolate points of the space $\Psi(\mathcal{M})$), but it is not submetrizable space.

Theorem 4.2. *Let X be a Tychonoff space, $\lambda \in \Psi$. Then following conditions are equivalent:*

1. *$C_\lambda(X)$ is a separable space;*
2. *X is a separably λ-submetrizable space.*

Proof. (1) $\Rightarrow$ (2). Let D be a countable dense subset of $C(X)$. Let us observe that for the diagonal map $f = \triangle_{f_i \in D} f_i$ we have $f : X \mapsto Y$, where $Y = f(X) \subseteq \prod\limits_{i \in \mathbb{N}} \mathbb{R}_i$ and f is one-to-one on $\bigcup \lambda$.

(2) $\Rightarrow$ (1). Consider the set $C(f(X))$ with the $f(\lambda)$-open topology. Note that $f(\lambda)$ is the family of compact subsets of $f(X)$ and it is closed under compact subsets of its elements. By Theorem (N. Noble, [18]), the space $C_c(f(X))$ is separable, so that $C_{f(\lambda)}(f(X))$ is separable too. It follows immediately that $C_\lambda(X)$ is a separable space. □

Corollary 4.3. *Let X be a Tychonoff space, $\lambda \in \Psi$ and let $C_\lambda(X)$ be a separable space. Then any element of the family λ is a metrizable compact subset of X.*

Theorem 4.4. *Let X be a Tychonoff separably λ-submetrizable space, λ, $\mu \in \Psi$ and $\mu \subseteq \lambda$. Then the following are equivalent:*

1. *$X \in S_1(\Lambda(\lambda), \Lambda(\mu))$;*
2. *$(C(X), \tau_\mu, \tau_\lambda)$ is $R_{(\tau_\lambda, \tau_\mu)}$-separable.*

Proof. (1) $\Rightarrow$ (2). By Theorem 4.2, $C_\lambda(X)$ is separable. On the other hand, by Theorem 3.7, the bispace $(C(X), \tau_\lambda, \tau_\mu)$ has countable (τ_λ, τ_μ)-strong fan tightness if (and only if) $X \in S_1(\Lambda(\lambda), \Lambda(\mu))$. By Corollary 3 in [15], we obtain $(C(X), \tau_\mu, \tau_\lambda)$ is $R_{(\tau_\lambda, \tau_\mu)}$-separable.

(2) $\Rightarrow$ (1). Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of λ-covers of X. For every $n \in \mathbb{N}$ let $A_n = \{f \in C_\lambda(X)$: there is $U \in \mathcal{U}_n, f(X \setminus U) = \{1\}\}$.

First, we verify that each A_n is dense in $C_\lambda(X)$. Let us consider $f \in C_\lambda(X)$ and let $\bigcap_{i \leq m} [K_i, V_i]$ be a basic neighborhood of f. The set $K = \bigcup_{i \leq m} K_i$ is compact and $K \in \lambda$. Since $\mathcal{U}_n$ is a λ-cover, there is $U \in \mathcal{U}_n$ containing K. There is also $g \in C_\lambda(X)$ such that $g(X \setminus U) = \{1\}$ and $g \upharpoonright K = f \upharpoonright K$. Then $g \in \bigcap_{i \leq m} [K_i, V_i] \bigcap A_n$.

Since $(C(X), \tau_\mu, \tau_\lambda)$ is $R_{(\tau_\lambda, \tau_\mu)}$-separable there are functions $f_n \in A_n$, $n \in \mathbb{N}$, such that the set $\{f_n : n \in \mathbb{N}\}$ is dense in $C_\mu(X)$. Let $U_n \in \mathcal{U}_n$ be a set for which $f_n(X \setminus U_n) = \{1\}$ holds. We claim that $\{U_n : n \in \mathbb{N}\} \in \Lambda(\mu)$. Let $F \in \mu$. Suppose that there is a point $x_n \in F \setminus U_n$ for each $n \in \mathbb{N}$. This means $f_n(x_n) = 1$ and it contradicts the fact that $\{f_n : n \in \mathbb{N}\}$ is dense in $C_\mu(X)$. □

Corollary 4.5. *Let X be a Tychonoff separably λ-submetrizable space, $\lambda \in \Psi$. Then the following are equivalent:*

1. *$X \in S_1(\Lambda(\lambda), \Lambda(\lambda))$;*
2. *$C_\lambda(X)$ is R-separable.*

Theorem 4.6. *Let X be a Tychonoff separably λ-submetrizable space, λ, $\mu \in \Psi$ and $\mu \subseteq \lambda$. Then the following are equivalent:*

1. $X \in S_{fin}(\Lambda(\lambda), \Lambda(\mu))$;
2. $(C(X), \tau_\mu, \tau_\lambda)$ *is* $M_{(\tau_\lambda,\tau_\mu)}$*-separable.*

Proof. (1) ⇒ (2). The space $C_\lambda(X)$ is separable since X is a separably λ-submetrizable space. On the other hand, by Theorem 3.11, (τ_λ, τ_μ)-fan tightness of $(C(X), \tau_\mu, \tau_\lambda)$ is countable if and only if X satisfies the selection property $S_{fin}(\Lambda(\lambda), \Lambda(\mu))$ and now apply Corollary 6 in [15].

(2) ⇒ (1). Assume that $(\mathcal{U}_n : n \in \mathbb{N})$ is a sequence of λ-covers. For every $n \in \mathbb{N}$ we set

$$V_n = \{f \in C_\lambda(X)\text{: there is } U \in \mathcal{U}, f(X \setminus U) = \{1\}\}.$$

We follow the proof of Theorem 4.4 to show that each V_n is dense in $C_\lambda(X)$.

Since $(C(X), \tau_\mu, \tau_\lambda)$ is $M_{(\tau_\lambda,\tau_\mu)}$-separable, there are finite sets $W_n = \{f_{n,1}, ..., f_{n,m_n}\} \subset V_n$, $n \in \mathbb{N}$, such that the set $\bigcup_{n\in\mathbb{N}} W_n$ is dense in $C_\mu(X)$.

Now consider the set $\mathbb{D}_n = \{U_{n,1}, ..., U_{n,m_n} : f_{n,i}(X \setminus U_{n,i}) = \{1\}, i \le m_n\}$ which is a finite subset of $\mathcal{U}_n$. It remains to show that $\bigcup_{n\in\mathbb{N}} \mathbb{D}_n \in \Lambda(\mu)$. Let $F \in \mu$. For some $j \in \mathbb{N}$ we have $[F, (-1, 1)] \cap W_j$, i.e. there is a function $f_{j,m_j} \in W_j$ such that $f_{j,m_j}(x) \in (-1, 1)$ for each $x \in F$. This means $F \subset U_{j,m_j}$ as required. □

Corollary 4.7. *Let X be a Tychonoff separably λ-submetrizable space, $\lambda \in \Psi$. Then the following are equivalent:*

1. $X \in S_{fin}(\Lambda(\lambda), \Lambda(\lambda))$;
2. $C_\lambda(X)$ *is M-separable.*

Recall that a bispace (X, τ_1, τ_2) is (τ_i, τ_j)-*Pytkeev* $(i \ne j; i, j = 1, 2)$ [13] (see also [23]) if for each $A \subset X$ and each $x \in Cl_i(A) \setminus A$ there are infinite sets $B_n \subset A$, $n \in \mathbb{N}$, such that each τ_j-neighborhood of x contains some B_n.

By Theorem 9 in [15], we have the following result.

Theorem 4.8. *Let X be a Tychonoff separably λ-submetrizable space, λ, $\mu \in \Psi$ and $\mu \subseteq \lambda$. If $(C(X), \tau_\mu, \tau_\lambda)$ is an $M_{(\tau_\mu,\tau_\lambda)}$-separable and (τ_μ, τ_λ)-Pytkeev bispace, then it is $R_{(\tau_\mu,\tau_\lambda)}$-separable.*

Corollary 4.9. *Let X be a Tychonoff separably λ-submetrizable space, $\lambda \in \Psi$. If $C_\lambda(X)$ is an M-separable and Pytkeev space, then it is R-separable.*

5. Bitopological H-separability and GN-separability

In this section we will be interested in some results on bitopological H-separability and GN-separability.

We begin by recalling the notion of weak selectively Reznichenko property for the bitopological spaces. A bispace (X, τ_1, τ_2) has the *weak selectively* (τ_i, τ_j)*-Reznichenko property* $(i \ne j; i, j = 1, 2)$, if for each sequence $(A_n : n \in \mathbb{N})$ of subsets of X and each point $x \in \bigcap_{n\in\mathbb{N}} Cl_i(A_n)$, there are finite sets $B_n \subset A_n$, $n \in \mathbb{N}$, such that each τ_j-neighborhood of x intersects B_n for all but finitely many n.

The definition of the selective bitopological version of the Reznichenko property was introduced (for hyperspace) in [14] and then considered in [23] and [24] in the function spaces context.

The notion of weak selectively Reznichenko property was introduced in [15]. The following results may be proved as Theorem 10 and Theorem 11 in [15].

Theorem 5.1. *Let X be a Tychonoff separably λ-submetrizable space, λ, $\mu \in \Psi$ and $\mu \subseteq \lambda$. Then the following are equivalent:*

1. $(C(X), \tau_\mu, \tau_\lambda)$ *is $H_{(\tau_\lambda,\tau_\mu)}$-separable;*
2. *For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of λ-covers there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of finite sets such that for each n, $\mathcal{V}_n \subset \mathcal{U}_n$ and each $F \in \mu$ is contained in an element of $\mathcal{V}_n$ for all but finitely many $n \in \mathbb{N}$.*

Proof. (1) $\Rightarrow$ (2). Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of λ-covers. For every $n \in \mathbb{N}$ let

$$D_n = \{f \in C_\lambda(X)\text{: there is } U \in \mathcal{U}_n, f(X \setminus U) = \{1\}\}.$$

One can easily prove that each D_n is dense in $(C(X), \tau_\lambda)$. Since $(C(X), \tau_\mu, \tau_\lambda)$ is $H_{(\tau_\lambda,\tau_\mu)}$-separable, there are finite sets $F_n \subset D_n$, $n \in \mathbb{N}$, such that each τ_μ-open set intersects F_n for all but finitely many n. Let $\mathcal{V}_n$, $n \in \mathbb{N}$, be the family of sets $U_f \in \mathcal{U}_n$, $f \in F_n$, such that $f(X \setminus U_f) = \{1\}$. We need to verify that the sequence $(\mathcal{V}_n : n \in \mathbb{N})$ witnesses that X satisfies (2).

Let $K \in \mu$. The open neighborhood $H = [K, (-1, 1)]$ intersects F_m for each m bigger than some m_0; now pick $f_m \in H \bigcap F_m$, $m > m_0$. Then for each $m > m_0$, $K \subset U_{f_m} \in \mathcal{V}_m$, as required in (2).

(2) $\Rightarrow$ (1). The proof consists of two parts.

Claim 1. *$C(X)$ has the weak selectively (τ_λ, τ_μ)-Reznichenko property.*

We take a sequence $(D_n : n \in \mathbb{N})$ of subsets of $C(X)$ whose closures contain $\mathbf{0}$. For every $T \in \lambda$ and every $m \in \mathbb{N}$ the τ_λ-neighborhood $[T, \frac{1}{m}]$ of $\mathbf{0}$ intersects each D_n. So, for each $n \in \mathbb{N}$ there exists a function $f_{T,n,m} \in D_n$ satisfying $|f_{T,n,m}(x)| < \frac{1}{m}$ for each $x \in T$. For each n set

$$\mathcal{U}_{n,m} = \{f^{-1}(-\frac{1}{m}, \tfrac{1}{m}) : m \in \mathbb{N}, f \in D_n\}.$$

(Given a bijection $\varphi : \mathbb{N}^2 \mapsto \mathbb{N}$ we put $\mathcal{U}_{n,m} := \mathcal{U}_{\varphi(m,n)}$.) We claim that for each $n, m \in \mathbb{N}$, each $C \in \lambda$ is contained in an element of $\mathcal{U}_{n,m}$. Indeed, if $C \in \lambda$, then there is $f_{C,n,m} \in [C, \frac{1}{m}] \bigcap D_n$. Hence $|f_{C,n,m}(x)| < \frac{1}{m}$ for each $x \in C$. This shows that $C \subset f^{-1}_{C,n,m}(-\frac{1}{m}, \frac{1}{m}) \in \mathcal{U}_{n,m}$.

Put $S := \{m \in \mathbb{N} : X \in \mathcal{U}_{n,m} \text{ for some } n \in \mathbb{N}\}$. There are two cases to consider.

Case 1. S is infinite.

There are $m_1 < m_2 < ...$ in S and $n_1, n_2, ...$ in $\mathbb{N}$, such that $f^{-1}_{T_i,n_i,m_i}(-\frac{1}{m_i}, \frac{1}{m_i}) = X$ for all $i \in \mathbb{N}$ and some $T_i \in \lambda$. Let $[R, \epsilon]$ be a τ_μ-neighborhood of $\mathbf{0}$. Pick m_k such that $\frac{1}{m_k} < \epsilon$. For every $m_i > m_k$ we have $f_{T_i,n_i,m_i}(x) \in (-\frac{1}{m_i}, \frac{1}{m_i})$ for each $x \in X$ and so $f_{T_i,n_i,m_i} \in [R, \frac{1}{m_i}] \subset [R, \epsilon]$. This means that the sequence $(f_{T_i,n_i,m_i} : i \in \mathbb{N})$ τ_μ-converges to $\mathbf{0}$, hence $C(X)$ has the weak selectively (τ_λ, τ_μ)-Reznichenko property at $\mathbf{0}$.

Case 2. S is finite.

There is $m_0 \in \mathbb{N}$ such that for each $m \geq m_0$ and each $n \in \mathbb{N}$, the set $\mathcal{U}_{n,m}$ is a λ-cover of X. We may suppose $m_0 = 1$. Further, we can consider only λ-covers $\mathcal{U}_{n,n}$, $n \in \mathbb{N}$. We can apply the condition (2) of this theorem to the sequence $\mathcal{U}_{n,m}$ to get a sequence $\mathcal{V}_{n,m}$, where for each $n \in \mathbb{V}$, $\mathcal{V}_{n,m}$ is a finite subset of $\mathcal{U}_{n,m}$ so that each $R \in \mu$ belongs to some $V \in \mathcal{V}_{n,n}$ for all but finitely many n. Choose the corresponding functions $f_{T_V,\frac{1}{n},\frac{1}{n}}$, $V \in \mathcal{V}_{n,n}$, and put $F_n = \{f_{T_V,\frac{1}{n},\frac{1}{n}} : V \in \mathcal{V}_{n,n}\}$. Then each F_n is a finite subset of D_n. Let $[R, \frac{1}{i}]$ be a neighborhood of $\mathbf{0}$. Let n_0 be such that $\frac{1}{n} < \frac{1}{i}$ and for each $n > n_0$ there is $V_n \in \mathcal{V}_{n,n}$ containing R. Choose a corresponding $f_n \in F_n$. Since this can be done for all $n > n_0$, we conclude that for all $n > n_0$ we have $f_n \in [R, \frac{1}{i}]$, i.e., $F_n \bigcap [R, \frac{1}{i}] \neq \emptyset$ for all $n > n_0$.

We now get back to proving the theorem.

Claim 2. *$C(X)$ is $H(\tau_\lambda, \tau_\mu)$-separable.*

Since X is a Tychonoff separably λ-submetrizable space, and $\tau_\mu \leq \tau_\lambda$, there is a countable dense subset $D = \{d_n : n \in \mathbb{N}\}$ in $(C(X), \tau_\lambda)$, so also in $(C(X), \tau_\mu)$. Let $(E_n : n \in \mathbb{N})$ be a sequence of dense subsets of $(C(X), \tau_\lambda)$. Fix $m \in \mathbb{N}$. Since $d_m \in Cl_{\tau_\lambda}(E_n)$ for each $n \in \mathbb{N}$, and $C(X)$ has the weak selectively (τ_λ, τ_μ)-Reznichenko property, there are finite sets $R_{n,m}$, such that for each n, $R_{n,m} \subset E_n$ and each τ_μ-neighborhood of d_m intersects all but finitely many $R_{n,m}$. For each n put $R_n = \bigcup\{R_{n,m} : m \leq n\}$. The sequence $(R_n : n \in \mathbb{N})$ witnesses for $(E_n : n \in \mathbb{N})$ that $C(X)$ is $H_{(\tau_\lambda, \tau_\mu)}$-separable. Indeed, let G be an open set in $(C(X), \tau_\mu)$. Then there is $d_m \in G$, hence G meets all but finitely many R_n. $\square$

Corollary 5.2. *Let X be a Tychonoff separably λ-submetrizable space, $\lambda \in \Psi$. Then the following are equivalent:*

1. *$C_\lambda(X)$ is H-separable;*
2. *For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of λ-covers there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of finite sets such that for each n, $\mathcal{V}_n \subset \mathcal{U}_n$ and each $F \in \lambda$ is contained in an element of $\mathcal{V}_n$ for all but finitely many $n \in \mathbb{N}$.*

Theorem 5.3. *If $(C(X), \tau_\mu, \tau_\lambda)$ is $GN_{(\tau_\lambda, \tau_\mu)}$-separable, then X satisfies $S_{fin}(\Lambda(\lambda), \Lambda(\mu)^{gp})$.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of λ-covers of X. Now we define sets D_n in $(C(X), \tau_\lambda)$ as in the proof of Theorem 5.1. These sets are d-dense in $C(X)$. Apply the fact that $C(X)$ is $GN_{(\tau_\lambda, \tau_\mu)}$-separable, so there are $f_n \in D_n$, $n \in \mathbb{N}$, such that $D = \{f_n : n \in \mathbb{N}\}$ is τ_μ-groupable, i.e. $D = \bigcup_{m \in \mathbb{N}} G_m$, where each $G_m = \{f_m^{k_1}, ..., f_m^{k_m}\}$ is a finite subset of D and each τ_μ-open set meets all but finitely many G_m. For each $m \in \mathbb{N}$, let

$$\mathcal{V}_m = \{U_m^{k_i} : f_m^{k_i}(X \setminus U_m^{k_i}) = \{1\}, i \leq m\}.$$

Let us show that each $F \in \mu$ is contained in some $V \in \mathcal{V}_m$ for all but finitely many m. Let $F \in \mu$. Then the τ_μ-open set $[F, 1]$ intersects G_m for all m bigger than some $m_0 \in \mathbb{N}$. Let $f_m^{k_j} \in [F, 1] \bigcap G_m$, $m \geq m_0$. Then $F \subseteq U_m^{k_j}$, $m \geq m_0$. It shows that X has $S_{fin}((\Lambda(\lambda), \Lambda(\mu)^{gp})$. $\square$

Corollary 5.4. *If $C_\lambda(X)$ is GN-separable, then X satisfies $S_{fin}(\Lambda(\lambda), \Lambda(\lambda)^{gp})$.*

By Theorem 3.7, Theorem 5.3 and Corollary 5.4 we obtain

Theorem 5.5. *If $(C(X), \tau_\mu, \tau_\lambda)$ is $GN_{(\tau_\lambda, \tau_\mu)}$-separable, then X satisfies $S_1(\Lambda(\lambda), \Lambda(\mu))$ and $S_{fin}(\Lambda(\lambda), \Lambda(\mu)^{gp})$.*

Corollary 5.6. *If $(C(X), \tau_\mu, \tau_\lambda)$ is $GN_{(\tau_\lambda, \tau_\mu)}$-separable, then $(C(X), \tau_\mu, \tau_\lambda)$ is $R_{(\tau_\lambda, \tau_\mu)}$-separable as well as $H_{(\tau_\lambda, \tau_\mu)}$-separable.*

Corollary 5.7. *If $C_\lambda(X)$ is GN-separable, then $C_\lambda(X)$ is R-separable as well as H-separable.*

6. Examples

We now consider some examples of the different type of separability in the bitopological function spaces.

Example 6.1. Let $\mathbb{I} = [0, 1] \subset \mathbb{R}$.

• By Example 2.14 in [4], $C_p(\mathbb{I})$ is M-separable, i.e. $\mathbb{I}$ has the property $S_{fin}(\Lambda(p), \Lambda(p))$ by Corollary 4.7. Since each k-cover of $\mathbb{I}$ is an ω-cover, we have that $\mathbb{I} \in S_{fin}(\Lambda(k), \Lambda(p))$. Hence the space $(C(\mathbb{I}), \tau_p, \tau_k)$ is $M_{(\tau_k,\tau_p)}$-separable.

• By Proposition 61 in [3], $C_p(\mathbb{I})$ is not R-separable, and, by Fact 2.1 in [15], the space $(C(\mathbb{I}), \tau_p, \tau_k)$ is not $R_{(\tau_k,\tau_p)}$-separable.

It follows that the bitopological space $(C(\mathbb{I}), \tau_p, \tau_k)$ is $M_{(\tau_k,\tau_p)}$-separable, but it is not $R_{(\tau_k,\tau_p)}$-separable.

Example 6.2. Let $X = \omega^\omega$. Then X is p-Lindelöf space, but $C_p(X)$ is not M-separable.

Recall that $\mathfrak{b}$ denote the minimum of cardinality of an unbounded set in ω^ω [7].

Example 6.3. Let X be an uncountable, second countable space of cardinality less than $\mathfrak{b}$. By Corollary 43 in [3], $C_p(X)$ is H-separable, but $C_p(X)$ is not R-separable, and, hence, $C_p(X)$ is not GN-separable.

Question 6.4. Does there exist a space X such that $(C(X), \tau_\mu, \tau_\lambda)$ is $R_{(\tau_\lambda,\tau_\mu)}$-separable and $H_{(\tau_\lambda,\tau_\mu)}$-separable, but it is not $GN_{(\tau_\lambda,\tau_\mu)}$-separable for some λ, $\mu \in \Psi$ and $\mu \subseteq \lambda$ (for $\lambda = k$ and $\mu = p$)?

Acknowledgements

The authors would like to thank the referees for their constructive comments and careful reading.

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